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Metastability and Exponential Approach to Equilibrium for Low-Temperature Stochastic Ising Models

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We present a proof of the exponential convergence to equilibrium of single-spin-flip stochastic dynamics for the two-dimensional Ising ferromagnet in the low-temperature case with not too small external magnetic field h uniformly in the volume and in the boundary conditions.

KEY WORDS: Ising model; Glauber dynamics; metastability.

1. INTRODUCTION AND MAIN RESULT

In this paper we prove a result on the convergence to equilibrium for some short-range single-spin-flip stochastic Ising models. In particular, we analyze the so-called "heat bath" algorithm and the "Metropolis" algorithm. These dynamics, described in detail in Section 2, satisfy the detailed balance condition with respect to the Gibbs measure of the Ising model and they are among the most popular dynamics used in Monte Carlo simulations of Ising models. A central question, both of practical and of theoretical interest, is the rate of approach to equilibrium. This question has been investigated in great detail by Holley in a series of remarkable papers[1,2] for general stochastic Ising models (see also Holley and

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Strook[3] and Aizenman and Holley[4]). In particular, Holley[1] proved that either the convergence is exponentially fast in time or else it is not faster, in a suitable sense, than t^{-d}, where d is the dimension of the underlying lattice. Moreover, Holley reduced the condition under which exponential convergence takes place to the validity of certain "mixing" conditions for the Gibbs measure in a finite region Λ similar to the famous Dobrushin–Shlosman uniqueness condition.[10] The region Λ on which the condition has to be valid is free and it must be conveniently chosen depending on the temperature and on the external magnetic field h. This approach allows one to prove exponential convergence for high temperature or for high magnetic field h. In both cases the region Λ may be chosen as consisting of very few sites. However, the interesting case of low temperature and small magnetic field remained open. The reason is that in Holley's approach the region Λ has to be chosen very large as h tends to zero, and already with regions consisting of few sites his conditions become intractable in practice. The physical reason is that, for the above values of the parameters (β, h), the Ising model exhibits a kind of asymmetric double-well structure with the deepest well representing the phase parallel to the magnetic field and the second well representing the metastable phase opposite to the field. Thus, a rigorous analysis of the convergence to equilibrium necessarily requires a detailed study of the transition (tunneling) from the metastable to the stable phase. This study has been done in great detail for the two-dimensional Ising model in finite volume in the zero-temperature limit by Jordão Neves and Schonman.[5] They proved in particular that the transition occurs essentially through the creation of a critical droplet of the right phase inside the metastable phase, and they derived rigorous upper and lower bounds on the typical time needed for this to happen.

In the present paper we combine some of the results of Holley together with the results of Jordão Neves and Schonman to provide a proof, entirely based upon an analysis of the dynamics, of the exponential convergence to equilibrium for the two-dimensional Ising model at low temperature and not too small magnetic field. The new ideas which enter in our proof are taken from an analogous work[6,7] (see also refs. 8, 9) done by the authors for the Swendsen–Wang dynamics in the same region of the parameters. The restriction here to the two-dimensional case is due to the fact that the results of ref. 5 are proved only in this case.

Let us state more precisely our main result. We consider the 2-dimensional nearest-neighbor Ising system in a cubic box

$$\Lambda = \{x \in \mathbf{Z}^2,\ x = (x_1, x_2)\colon |x_i| \leqslant L,\ i = 1, 2\}$$

of side $2L+1$ whose Hamiltonian is written as

$$H_\Lambda^b(\sigma) = -\frac{1}{2} \sum_{x,\, y \in \Lambda,\, |x-y|=1} \sigma(x)\,\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x) \text{ and}$$
$$-\frac{1}{2} \sum_{x \in \Lambda,\, z \notin \Lambda,\, |x-z|=1} \sigma(x)\, b(z)$$

where $\sigma \in \{-1, 1\}^\Lambda$, and b represents the boundary conditions.

Let

$$\mu_\Lambda = \exp(-\beta H_\Lambda^b)/Z_\Lambda$$

be the Gibbs measure in Λ with boundary conditions b, inverse temperature β, and external magnetic field h (Z_Λ is the partition function). Let $E_\sigma(f(\sigma_t))$ denote the expectation value of the observable f over the process σ_t starting at σ and evolving either with the heat bath or with Metropolis algorithm. The following theorem holds.

Theorem. Given $h>0$, one can find $\beta_0 = \beta_0(h)$ and $t_0(\beta, h)$ such that, for any $\beta > \beta_0$, one can find $m > 0$ independent of the volume Λ and of the b.c. b such that

$$\sup_{\sigma \in \{-1,1\}^\Lambda} |\mu_\Lambda(f) - E_\sigma(f(\sigma_t))| \leqslant C_f \exp(-mt), \qquad \forall t \geqslant t_0(\beta, h)$$

where f is an arbitrary local observable and C_f a numerical constant depending only on f.

We would like to emphasize that we prove the above result only under the hypotheses βh large. This is so in order to be able to use the results of ref. 5. The above result is, however, expected to be true in the more general case when β is fixed large enough and h is different from zero. However, the mechanism leading to the loss of memory and therefore to exponential convergence to equilibrium in the case β large but βh small should be more complicated than the one described here. This is also suggested by the observation made by Huse and Fisher[12] for the case $h=0$ and $+$ boundary conditions. They remarked that in this case the convergence to the $+$ phase, starting, e.g., from all plus spins, cannot be exponential in two dimension because of the presence in the $+$ phase of rare large droplets of minus spins. Such droplets, in the absence of an external positive magnetic field, disappear in a time proportional to their area, while their probability is only exponentially small in the length of their boundary. This led to the conclusion that the convergence to equilibrium should be a stretched exponential $\exp(-\sqrt{t})$ (see also Sokal and Thomas[13]).

Let us now explain the physical mechanism behind the proof of the main theorem in the presence of an external magnetic field h such that βh is large. If we start with two opposite configurations, one with all spins minus and the other with all spins plus, and we let them evolve together (see below), due to the presence of the external field, the negative configuration will become mostly plus in a time independent of the volume (nucleation time) (see Theorem 3), while the positive configuration will continue to be mostly positive. Thus, after the nucleation time the differences between the two configurations will consist of small islands well isolated one from the other. These islands, under the action of the common sea of plus spins and of the field h, will disappear in a rather short time of the order of the $\log(L)$, where L is the size of the box under consideration. Thus, the mechanism is essentially local.

The paper is organized as follows: in Section 2 we illustrate the strategy of the proof and we prove the main steps modulo some technical results, whose proof is given in Section 3.

2. PROOF OF THE THEOREM

As already mentioned, we restrict ourselves to a stochastic Ising model in 2 dimensions (see, e.g., ref. 1) with flip rate $C(x, \eta)$, $\eta \in \{-1, 1\}^{\Lambda}$, $\Lambda = [-L, L]^2 \cap \mathbf{Z}^2$, given by

$$C(x, \eta) = \{1 + \exp[\beta\, \Delta_x H(\eta)]\}^{-1}$$

for the heat bath (HB) and

$$C(x, \eta) = \begin{cases} 1 & \text{if } \ \Delta_x H(\eta) \leqslant 0 \\ \exp\{-\beta\, \Delta_x H(\eta)\} & \text{otherwise} \end{cases} \tag{2.1}$$

for the Metropolis (M) algorithm, where $H(\eta)$ is the Ising Hamiltonian:

$$H(\eta) = -\frac{1}{2} \sum_{\substack{\langle xy \rangle \\ \varepsilon \Lambda}} \eta(x)\, \eta(y) - \frac{1}{2} h \sum_{x \in \Lambda} \eta(x), \qquad h > 0 \tag{2.2}$$

and

$$\Delta_x H(\eta) = H(\eta^x) - H(\eta) \tag{2.3}$$

with $\eta^x(y) = \eta(y)$ if $x \neq y$, $\eta^x(x) = -\eta(x)$. Here we are considering for simplicity open boundary conditions; periodic, plus, or minus boundary conditions may be considered as well. For each initial configuration η each of the above rates defines a continuous-time Markov process denoted by $\{\eta_t\}_{t>0}$ such that

$$\eta_0(x)=\eta(x), \qquad \forall x \in \Lambda$$

$$P(\eta_{t+\varepsilon}=\sigma \mid \eta_t=\sigma_1)=\begin{cases} C(x, \sigma_1)\varepsilon + o(\varepsilon) & \text{if} \quad \sigma=(\sigma_1)^x \text{ for some } x \in \Lambda \\ o(\varepsilon^2) & \text{otherwise} \end{cases}$$

as $\varepsilon \to 0$.

An explicit construction of the process goes as follows

(a) With rate $|\Lambda|$ one chooses a random site $x \in \Lambda$. We will then say that "x is visited."

(b) Given x, one extracts a number

$$\xi_x \in (0, 1) \tag{2.4}$$

with uniform distribution.

(c) Given a configuration η, one sets

$$\begin{aligned} &\eta_\tau=\eta^x && \text{iff} \quad \xi_x < C(x, \eta) \\ &\eta_\tau=\eta && \text{otherwise} \end{aligned}$$

where τ is the (random) time at which x was visited.

The property which is crucial for our results is that each of the above flip rates defines an "attractive" stochastic Ising model; this means that if f is a monotone function in the sense that $f(\sigma) \geqslant f(\eta)$ for any pair σ, $\eta \in \{-1, 1\}^\Lambda$ with $\sigma(x) \geqslant \eta(x)$, $\forall x$, then $E_\eta f(\eta_t)$ is also monotone as function of η.

For the HB algorithm there is actually an alternative explicit construction of the process which preserves the partial ordering between configurations $\sigma \leqslant \eta$ *path by path*. This is realized by replacing (c) with:

(c′) Given a configuration η, one sets

$$\begin{aligned} &\eta_\tau(x)=+1 && \text{if} \quad \xi_x < \left(1+\exp\left\{-\beta\left[h+\sum_{|y-x|=1} \eta(y)\right]\right\}\right)^{-1} \\ &\eta_\tau(x)=-1 && \text{otherwise} \\ &\eta_\tau(y)=\eta(y) && \forall y \neq x \end{aligned}$$

It is clear that the above explicit construction of the process based on (a), (b), (c′) above provides a random flow ϕ_t on $\{-1, 1\}^\Lambda$ and by explicit computation one checks that

$$\begin{aligned} &\text{if} \quad \sigma \leqslant \eta, \quad \text{i.e.,} \quad \sigma(x) \leqslant \eta(x), \quad \forall x \in \Lambda \\ &\quad \text{then} \quad \phi_t(\sigma) \leqslant \phi_t(\eta) \end{aligned} \tag{2.5}$$

Thus, in this case if $\phi_t(\pm 1)$ denote the evolution at time t of the configurations identically equal to ± 1, we get

$$\phi_t(-1) \leqslant \phi_t(\sigma) \leqslant \phi_t(1), \qquad \forall \sigma \in \{-1, 1\}^\Lambda$$

This property clearly implies attractivity. Using this property, some of the proofs become more transparent; therefore we decided to provide all the details only for the HB algorithm constructed with (a), (b), (c′).

Moreover, in order to make the exposition easier, we assume $h \ll 1$. At the end of Section 3 we will sketch the simple modifications required by the M algorithm and the extension of the results to general $h > 0$. We now turn to the proof of our main theorem. It is one of the important results of Holley[1] that the theorem follows if we can prove:

Theorem 2. For any sufficiently large β there exists $t_\beta > c_0$ with the property that

$$\lim_{\beta \to +\infty} t_\beta^2 P(\phi_{t_\beta}(1)(0) \neq \phi_{t_\beta}(-1)(0)) = 0$$

The constant c_0 is a number independent of β and h (see Theorem of ref. 1).

A crucial ingredient in the proof of Theorem 2 is the following result, which is a rather direct consequence of the work by Jordão Neves and Schonmann.[5] For simplicity we assume, as in ref. 5, $2/h \notin \mathbb{N}$. Given an even number l, we set $Q^l \equiv \{\text{square centered at } (1/2, 1/2) \text{ with side } l\} \cap \mathbb{Z}^2$, $l > [2/h] + 1$ and

$$\tau(\sigma) = \inf\{t \geqslant 0; \phi_t(\sigma)(x) = +1, \forall x \in Q^l\}$$

Theorem 3. $\forall \delta > 0$,

$$\lim_{\beta \to +\infty} P(\tau(-1) < \exp[\beta(\Gamma(h) + \delta)]) = 1$$

where

$$\Gamma(h) = 4[2/h] - ([2/h]^2 + [2/h] + 1)h + 4$$

Remark 1. If we let $T_{n,\beta} \equiv \exp\{\beta(\Gamma(h) + n\delta)\}$, we get immediately from the strong Markov property that

$$P(\tau(-1) > T_{n,\beta}) \leqslant P(\tau(-1) > T_{1,\beta})^{[e^{(n-1)\delta\beta}]} \tag{2.6}$$

We have used the obvious fact that

$$\tau(-1) \geqslant \tau(\sigma), \qquad \forall \sigma \in \{-1, 1\}^{\Lambda}$$

We postpone the proof of Theorem 3 to Section 3 and we now prove Theorem 2.

Proof of Theorem 2. Let $t_\beta = T_{3,\beta}$, $\delta > 0$, and let Ω_0 be the event

$$\Omega_0 = \{\tau(-1) < e^{-\delta\beta} t_\beta\} \tag{2.7}$$

It follows from Theorem 3, Remark 1, and (2.5) that if $l > [2/h] + 1$ and $\beta \gg 1$,

$$t_\beta^2 P(\Omega_0^c) \leqslant \exp[-k(\beta)\exp(\delta\beta)] \tag{2.8}$$

with $k(\beta) \uparrow +\infty$ as $\beta \to +\infty$.

The idea is now very simple. We write

$$P(\phi_{t_\beta}(-1)(0) \neq \phi_{t_\beta}(+1)(0)) \leqslant P(\Omega_0^c) + P(\phi_{t_\beta}(-1)(0) \neq \phi_{t_\beta}(+1)(0); \Omega_0) \tag{2.9}$$

The first term, using (2.8), satisfies the condition of Theorem 2. The second term is given by

$$\int_0^{t_\beta e^{-\delta\beta}} E\chi(\tau(-1) \in dt_0)\, \chi(\phi_{t_\beta}(-1)(0) \neq \phi_{t_\beta}(+1)(0)) \tag{2.10}$$

Notice that $\phi_{\tau(-1)}(-1)(x) = \phi_{\tau(-1)}(+1)(x)$, $\forall x \in Q^l$, and both are $+1$.

We have then to show that for arbitrary configurations σ and η such that

$$\sigma(x) = \eta(x) = +1, \qquad \forall x \in Q^l$$

the possible differences between σ and η outside Q^l are not able to reach the origin within the time $t_\beta - t_0$ with a probability greater than $1 - p(\beta)$, with

$$t_\beta^2 p(\beta) \to 0 \qquad \text{as} \quad \beta \to +\infty$$

In other words, the large cluster Q^l of plus spins at time t_0 is able to completely screen the origin from the rest of the configuration outside Q^l

for a time longer than $t_\beta - t_0$. We are using here the rather trivial observation that, if, at time t, $\sigma_t(x) \neq \eta_t(x)$ and if $\bar{t}$ was the time of the last updating of the site x, then necessarily there exists a site y, $|y-x|=1$ with $\sigma_{\bar{t}-\delta}(y) \neq \eta_{\bar{t}-\delta}(y)$, $\forall$ sufficiently small δ.

This simply means that the differences between two configurations can propagate only between nearest neighbor sites and that they cannot be created from nothing.

In order to analyze more precisely the above-mentioned "screening" effect, we follow ref. 6. Let

$$I_j = \bigcup_{(j-1)l^2+1}^{jl^2} T_i$$

where $T_i = [(i-1)\beta^2, i\beta^2]$, and let $v_i(x)$ be the number of times the site x is visited during T_i.

Definition 1. I_j is "good" if:

(i) $\forall x \in Q^l$, $\forall i \in [(j-1)l^2, jl^2] \cap \mathbf{N}$,

$$1 \leqslant v_i(x) \leqslant 2\beta^2 \tag{2.11}$$

(ii) $\forall x \in Q^l$, $\forall i \in [(j-1)l^2, jl^2] \cap \mathbf{N}$, $1-\exp(-\beta h) > \xi_x^k > \exp(-\beta h)$, where ξ_x^k is the random number in $(0, 1)$ that one extracts in order to update x the kth time in the time interval T_i.

Remark 2. Point (ii) of (2.11) ensures that, uniformly in the initial configuration σ, during a "good" time interval I_j we never see an updating in Q^l which increases the energy.

Definition 2:

(a) I_j is "bad" if it is not "good."

(b) $S_n \equiv \bigcup_{j=j_0}^{j_0+n} I_j$ is called a "bad" (maximal) sequence if I_j is "bad" $\forall j \in (j_0, j_0+n)$ and I_{j_0-1}, I_{j_0+n+1} are "good."

(c) A "bad" sequence S_n is said to be "acceptable" iff $\#\{t \in S_n$; there is $x \in Q^l$ such that x is visited at time t with $\xi < \exp(-\beta h)$ or $\xi > 1-\exp(-\beta h)\} < 20/h^2$.

The main reason for the introduction of the above definitions is explained in the next two propositions, which will be proven in Section 3.

Proposition 1. Let σ be such that there exists a subset $A \subset Q^l$ with $\sigma(x) = +1$, $\forall x \in A$, and with the property that:

(1) the smallest rectangle enclosing A coincides with Q^l,

(2) there exists no $x \in A$ such that $\sum_{|y-x|=1} \sigma(y) < 0$.

Then if I_1 is "good," we have

$$\sigma_{l^2\beta^2}(x) = 1 \qquad \forall x \in Q^l$$

In other words, a "good" interval of time is able to reconstruct the full square Q^l of plus spins provided that one starts with a cluster of plus spins which is only a "small erosion" of Q^l.

For brevity we will refer to conditions 1 and 2 of the above proposition as property $\mathscr{P}$.

Proposition 2. Let σ be such that $\sigma(x) = +1$, $\forall x \in Q^l$, with $l > 2/h^3$, and let $S_n = I_1 \cup \cdots \cup I_n$ be "acceptable." Then, if h is small enough, with probability one for any $t \in S_n$ there exists a subset $\Lambda_t^+ \subset Q^l$ with the property $\mathscr{P}$ such that $\sigma_t(x) = +1$, $\forall x \in \Lambda_t^+$, $|\Lambda_t^+| > l^2 - l/2$.

We can say that an "acceptable" sequence S_n can only produce "small erosion" (in the sense explained above) of an initial cluster Q^l of plus spins provided l is large enough. Thus, if we have a time interval $[0, T] = \bigcup_{j=1}^N I_j$ such that any bad sequence S_n in $[0, T]$ is "acceptable" and if $\sigma \in \{-1, 1\}^\Lambda$ is such that $\sigma(x) = +1$, $\forall x \in Q^l$, then in $[0, T]$ the cluster of plus spins inside Q^l undergoes only small fluctuations around its starting shape Q^l.

More important, if σ and η are such that $\sigma(x) = \eta(x) = +1$, $\forall x \in Q^l$, then, since $|\Lambda_t^+| > l^2 - l/2$, we will never see a difference at $x = 0$ within time T, i.e.,

$$\sigma_t(0) = \eta_t(0), \qquad \forall t \in [0, T] \tag{2.12}$$

Thus

$$\sup_{\sigma, \eta: \eta(x) = \sigma(x) = 1, \forall x \in Q^l} \; \sup_{t_0 < e^{-\delta\beta} t_\beta} P(\phi_{t_\beta - t_0}(\sigma)(0) \neq \phi_{t_\beta - t_0}(\eta)(0))$$

can be bounded by

$$\begin{aligned} &P(\exists \text{ a "nonacceptable" sequence } S_n \text{ in } [t_0, t_\beta]) \\ &\quad \leqslant P(\exists \text{ a "nonacceptable" sequence } S_n \text{ in } [0, t_\beta]) \\ &\quad \leqslant \frac{t_\beta}{l^2\beta^2} \sum_{n=1}^{\infty} P(S_n = I_1 \cup \cdots \cup I_n \text{ is bad and "nonacceptable"}) \end{aligned} \tag{2.13}$$

The generic term of the sum in the rhs of (2.13) is estimated in the final proposition.

Proposition 3. If β is large enough,

$$P(S_n \text{ is a bad "nonacceptable" sequence}) \leqslant e^{-\beta hn/2} \qquad \text{if} \quad n > \frac{30}{h^2}$$

$$P(S_n \text{ is a bad "nonacceptable" sequence}) \leqslant e^{-19\beta/h} \qquad \text{if} \quad n < \frac{30}{h^2}$$

Using the above proposition, we get that the rhs of (2.13) is bounded by

$$\frac{t_\beta 2e^{-15\beta/h}}{l^2\beta^2} + \frac{t_\beta}{l^2\beta^2}\frac{30}{h^2}e^{-19\beta/h} \equiv p(\beta) \tag{2.14}$$

It follows from the definition of t_β that

$$p(\beta)t_\beta^2 \to 0 \qquad \text{as} \quad \beta \to +\infty \tag{2.15}$$

and Theorem 2 is proved. ▮

3. SOME TECHNICAL PROOFS

This section contains the proofs of Theorem 3 and Propositions 1–3.

Proof of Theorem 3. Given Q^l, $l > [2/h] + 1$, let ∂_l be the external boundary of Q^l, i.e.,

$$\partial_l = \{x \notin Q^l, \exists y \in Q^l, |x - y| = 1\} \tag{3.1}$$

We want to construct the dynamics on $\{-1, 1\}^\Lambda$ with extra minus boundary conditions on ∂.

This is done by adding to the step (c′) of the explicit construction of our process (see Section 2) the condition

$$\eta_\tau(y) = -1, \qquad \forall y \in \partial_l \tag{3.2}$$

The new random flow on $\{-1, 1\}^\Lambda$ is denoted by $\phi_t^\partial(\cdot)$. It is easy to check that

$$\phi_t^\partial(\sigma)(x) \leqslant \phi_t(\sigma)(x), \qquad \forall x \in \Lambda, \quad \forall \sigma \in \{-1, 1\}^\Lambda \tag{3.3}$$

Therefore, if we let

$$\tau^\partial(\sigma) = \inf\{t \geqslant 0; \phi_t^\partial(\sigma)(x) = +1, \forall x \in Q^l\} \tag{3.4}$$

then we have $\tau^\partial(\sigma) \geqslant \tau(\sigma)$, $\forall \sigma \in \{-1, 1\}$. Since the regions Q^l and $\Lambda \backslash (Q^l \cup \partial)$ are now decoupled for the dynamics $\phi_t^\partial(\cdot)$, we have reduced the original problem to an analogous problem on a finite volume Q^l.

Shonmann and Jordão Neves[5] proved the result of Theorem 3 for the Metropolis dynamics on Q_φ^l with periodic boundary conditions. It is easy to extend their arguments to cover the case of heat bath dynamics on Q_φ^l with minus boundary conditions. ∎

Proof of Proposition 1. It is quite clear that, since I_1 is good, i.e., there is no updating in Q^l during I_1 which increases the energy, $\sigma_t(x) = +1$, $\forall x \in A$, $\forall t \in I_1$.

Let now y be a nearest neighbor of A such that $\exists x$, $x_1 \in A$ with $|x-y| = |x_1 - y| = 1$. Such a y exists unless A coincides with Q^l.

Let τ_y be the first time in T_1 for which the site y is visited. τ_y exists since I_1 is good. Then, if $\sigma_t(y)$ was not already $+1$ for $t < \tau_y$, at time τ_y it flips to $+1$ in order to lower the energy.

We then set

$$A_1 = A \cup y$$

Clearly, A_1 enjoys property $\mathscr{P}$ and we can repeat the above argument for A_1. We iterate the above procedure until we have reconstructed all Q^l at some $\bar{t} \in I_1$. Once the droplet Q^l is formed with all spins equal to $+1$, then it is stable, since I_1 is good and the proposition follows. ∎

Proof of Proposition 2. The proof consists essentially in bounding the total loss of plus spins inside Q^l at the end of the "bad" sequence S_n. We first define inductively the random time

$$\tau_k, \qquad k = 0, 1,..., k_0, k_0 < 20/h^2$$

by

$$\tau_0 = 0$$

$$\tau_k = \inf\{t \geqslant \tau_{k-1}, \exists x \in Q_0^l \text{ such that } x \text{ is visited at time } t \text{ and } \xi_x < \exp(-\beta h)\} \tag{3.5}$$

By construction, between (τ_{k-1}, τ_k) there is no updating inside Q^l which increases the energy $H(\sigma_t)$.

Let us now partition the square Q^l into $(l/2)^2$ squares $\bar{Q}_j$ of side 2. By construction, if at time $\tau_k + \delta$ all the spins inside $\bar{Q}_j$ are plus one for all δ sufficiently small, then each one of them will not flip up to time τ_{k+1}. Moreover, at time τ_k there exists at most one square $\bar{Q}_j$ such that

$$\sigma_{\tau_k - \delta}(x) = +1 \qquad \forall x \in \bar{Q}_j \tag{3.6}$$

$$\sigma_{\tau_k + \delta}(y) = -1 \qquad \text{for some } y \in \bar{Q}_j \tag{3.7}$$

if δ is small enough.

If we denote by J the set of all indices j such that the 2×2 square $\bar{Q}_j$ has never been visited for any τ_k with

$$\tau_k \leqslant t \tag{3.8}$$

we define

$$\Lambda_t^+ = \bigcup_{j\in J} \bar{Q}_j \tag{3.9}$$

Evidently Λ_t^+ enjoys the property $\mathscr{P}$ and

$$|\Lambda_t| \geqslant l^2 - \frac{4\cdot 20}{h^2} \tag{3.10}$$

$$\frac{4\cdot 20}{h^2} < \frac{l}{2} \tag{3.11}$$

if h is small enough and $l > 1/h^3$.

Thus, the smallest rectangle in Q^l containing Λ_t^+ is Q^l and the proposition follows. ▮

Proof of Proposition 3. We first compute the probability of a "bad" interval I_1. We have

$$P(I_1 \text{ is "bad"}) \leqslant e^{-\beta h/2} \tag{3.12}$$

if βh is large enough.

In fact

$$\begin{aligned} &P(I_1 \text{ is "bad"}) \\ &\quad \leqslant l^4 \sup_{x\in Q^l} P(v_1(x) \notin (1, 2\beta^2)) \\ &\qquad + l^4 \sup_{x\in Q^l} P(v_1(x) \in (1, 2\beta^2); \\ &\qquad \exists k \leqslant v_1(x),\ \xi_x^k < \exp(-\beta h) \text{ or } \xi_x^k > 1 - \exp(-\beta h)) \end{aligned} \tag{3.13}$$

The first term, after an explicit computation, is bounded by $\exp(-\beta^2/2)$ for β sufficiently large, while the second one can be estimated by

$$(2l)^2 l^2 2\beta^2 e^{-\beta h} < \frac{e^{-\beta h/2}}{2}$$

and (3.12) follows.

Thus we have either the bound

$$P(S_n \text{ is a bad "nonacceptable" sequence}) \leqslant e^{-\beta h n/2} \tag{3.14}$$

or:

$$\begin{aligned}
&P(S_n \text{ is a bad "nonacceptable" sequence})\\
&\quad\leqslant P\Bigg(\#\{t \in S_n;\ \exists x \in Q_0^l \text{ visited at time } t\\
&\qquad\text{with } \xi_x < \exp(-\beta h) \text{ or } \xi_x > 1-\exp(-\beta h)\} > \frac{20}{h^2}\Bigg)\\
&\quad\leqslant e^{-19\beta/h} \qquad \text{for } \beta \text{ large enough if } n < 30/h^2
\end{aligned} \tag{3.15}$$

The last bound follows from a trivial estimate on the independent process of choosing a site x with rate $|\Lambda|$.

The proposition is proved. ∎

We now sketch the modifications of the above argument necessary to cover the M case, since for the random flow $\tilde{\phi}_t$ given by (a), (b), (c) it is not true anymore that

$$\tilde{\phi}_t(\sigma) \leqslant \tilde{\phi}_t(\eta) \qquad \text{if} \quad \sigma < \eta$$

We start with Theorem 3 and we proceed as in the HB dynamics. Minus boundary conditions on ∂ [see (3.1)] can be introduced as in (3.2). It turns out that the corresponding flip rates $C^{\partial}(x, \eta)$ are such that

$$\begin{aligned}
C^{\partial}(x, \eta) \geqslant C(x, \sigma) \qquad &\text{if } \eta \leqslant \sigma \text{ and } \sigma(x) = \eta(x) = +1\\
C^{\partial}(x, \eta) \leqslant C(x, \sigma) \qquad &\text{if } \eta \leqslant \sigma \text{ and } \sigma(x) = \eta(x) = -1
\end{aligned} \tag{3.16}$$

We can therefore apply Theorem 1.5, Chapter 3 of ref. 11 to get

$$\sup_{\sigma} P(\tau(\sigma) \geqslant e^{-\partial\beta} t_\beta) \leqslant \sup_{\sigma} P(\tau^{\partial}(\sigma) > e^{-\delta\beta} t\beta) \tag{3.17}$$

where $\tau(\sigma)$, $t^{\partial}(\sigma)$, t_β are defined as in Section 2 and in (3.4).

The rhs of (3.17), by the Markov property, can be bounded by

$$\sup_{\sigma} P(\tau^{\partial}(\sigma) > T_{1,\beta})^{[e^{\delta\beta}]} \tag{3.18}$$

where $T_{1,\beta}$ is as in Theorem 3.

Thus, as in the HB dynamics, thanks to the results of ref. 5, with very large probability $\tau(-1)$ and $\tau(+1)$ are both less than $e^{-\delta\beta} t_\beta$. Notice, however, that since $\tilde{\phi}_t(\cdot)$ does not preserve ordering, it is no longer true in general that

$$\tilde{\phi}_{\tau(-1)}(+1)(x) = +1, \qquad \forall x \in Q^l$$

As in the HB case, one then defines the good intervals and the acceptable sequences using only the basic underlying process which chooses the random site x and the random numbers ξ_x.

Thus, Propositions 1–3 remain unchanged. If now the time interval $[0, t_\beta]$ contains only acceptable sequences, by Propositions 1 and 2, at time t, the end of the first good interval I_j after $\tau(-1) \vee \tau(+1)$, both $\tilde{\phi}_t(-1)$ and $\tilde{\phi}_t(+1)$, will be identically equal to $+1$ in Q^l. After this step the proof is unchanged.

We finally describe the extension of our results to arbitrary $h > 0$.

Let $h_2 > 0$ and let $0 < h_1 < h_2$ be so small that the previous results apply for $h = h_1$. It is easy to check that if we denote by $C_1(x, \eta)$, $C_2(x, \eta)$ the flip rates for $h = h_1$, $h = h_2$, respectively, then C_1, C_2 satisfy

$$\begin{aligned} C_1(x, \eta) &\geqslant C_2(x, \sigma) \qquad \text{if} \quad \eta \leqslant \sigma, \quad \eta(x) = \sigma(x) = 1 \\ C_1(x, \eta) &\leqslant C_2(x, \sigma) \qquad \text{if} \quad \eta \leqslant \eta(x) = \sigma(x) = -1 \end{aligned} \tag{3.19}$$

Thus, again using Theorem 1.5, Chapter 3, of ref. 11,

$$\sup_\sigma P(\tau_2(\sigma) \geqslant t) \leqslant \sup_\sigma P(\tau_1(\sigma) \geqslant t)$$

where $\tau_1(\sigma)$, $\tau_2(\sigma)$ are $\tau(\sigma)$ with $h = h_1$, h_2, respectively.

It follows that, if we define for $h = h_2$ the good time intervals, the acceptable sequences, and the time t_β to be exactly those define for $h = h_1$, the previous proof applies to $h = h_2$ as well. Of course, in this way the time t_β will not coincide in general with the "true time" of $h = h_2$.

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